Ballooning theory of the second kind—two dimensional tokamak modes

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The 2-D ballooning transform, devised to study local high toroidal number (n) fluctuations in axisymmetric toroidal system (like tokamaks), yields a well-defined partial differential equation for the linear eigenmodes. In this paper, such a ballooning equation of the second kind is set up for ion temperature gradient driven modes pertinent to a 2-D non-dissipative fluid plasma; the resulting partial differential equation is numerically solved, to calculate the global eigenvalues, and the 2-D mode structure is presented graphically along with analytical companions. The radial localization of the mode results from translational symmetry breaking for growing modes and is a vivid manifestation of spontaneous symmetry breaking in tokamak physics. The eigenmode, poloidally ballooned at \( \theta = \pm \pi/2 \), is radially shifted from associated rational surface. The global eigenvalue is found to be very close to the value obtained in 1-D parameterized (\( \lambda = \mp \pi/2 \)) case. The 2-D eigenmode theory is applied to estimate the toroidal seed Reynolds stress [Y. Z. Zhang, Nucl. Fusion Plasma Phys. 30, 193 (2010)]. The solution obtained from the relatively simplified ballooning theory is compared to the solution of the basic equation in original coordinate system (evaluated via FFTs); the agreement is rather good. © 2012 American Institute of Physics. [http://dx.doi.org/10.1063/1.4731724]

I. INTRODUCTION

The late 1970s witnessed the emergence of the conventional ballooning transform (the ballooning theory of first kind (BM-I)) as a tool to solve the 2-D local eigenvalue problem for high toroidal number (n) in axisymmetric toroidal system like tokamaks.1–3 The new methodology reduced the intrinsic 2-D eigenmode equation to a much simpler 1-D ballooning equation; the feat was accomplished by exploiting the approximate (to the lowest order in the small parameter \( 1/\sqrt{n} \)) translational invariance or the “ballooning symmetry” of the original system. In this formalism, the 2-D toroidal mode, radially localized at a rational surface, is also found to be poloidally localized at either \( \theta = 0 \) or \( \theta = \pi \). As a result, it admits the up-down symmetry in the poloidal cross section. However, the BM-I analysis is valid only if a well-defined solvability condition is satisfied, and this condition puts restrictions on the mode structure that was, initially, believed to exist only at a given rational position.4 For a complex system (including non-dissipative ones away from marginal stability (throughout this paper, the complex BM-II always refers to including the non-dissipative system having finite growth rate as well as the dissipative system), the solvability condition consists of two equations and results in a more stringent constraint for the occurrence of BM-I.5,6

While the existence of another type of solution different from BM-I was anticipated since early 1980s,7,8 the ballooning theory of the second kind (BM-II) was developed only in early 1990s.9–16 The main features of BM-II, in contrast to BM-I, can be summarized as follows: (a) no restriction due to solvability condition implying that the 2-D toroidal mode may occur at all radii; (b) the poloidal structure can only be localized at \( \theta = \pi/2 \) (or \( \theta = -\pi/2 \)); and (c) no up-down symmetry in the poloidal plane.

Both BM-I and BM-II begin by subjecting the original mode equation to the 2-D ballooning transform5

\[
\phi_1(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_{-\infty}^{+\infty} d\lambda \int_{-\infty}^{+\infty} dk e^{i(k(x-\lambda)-i\lambda^2)} \varphi(k,\lambda),
\]

(1)
a mathematically proper transform with a unique inverse. The LHS of Eq. (1), \( \phi_1(x) \), denotes the mode in the physical representation \((x,\lambda)\) where \( x \) is the radial variable and \( \lambda \) labels the discrete poloidal variable (see Eqs. (6) and (8) for detail). The variables \( x \) and \( \lambda \) enter the exponent as \( x-\lambda \) implying the translational invariance first noted by Lee and Van Dam.5 The transformed potential \( \phi(k,\lambda) \) is a function of two continuous variables \( k \in (-\infty, \infty) \) and \( \lambda \in [-\pi, \pi] \); the second variable \( \lambda \) “extends” the parameter \( \lambda(\theta_0) \) in Lee-Van Dam (Connor, Hastie, and Taylor) representation. Both these representations are special cases in which \( \phi(k,\lambda) \) is a Dirac \( \delta \)-function \( \delta(\lambda - \lambda_0) \). The explicit \( \lambda \)-dependence of \( \phi(k,\lambda) \) reminds us that the localization in \( \lambda \), the second dimension, is to be determined (or no localization at all as in the toroidal Alfvén eigenmode mode9,13,17–20) a post-priori as the solution of the partial differential equation.

Invoking Eq. (1), the large n, 2-D mode equation (for \( \phi_1(x) \)) transforms to the 2-D ballooning equation

\[
\left( L_0 + \frac{iL_1}{n} \frac{\partial}{\partial \lambda} + \frac{L_2}{n^2} \frac{\partial^2}{\partial \lambda^2} + \cdots - \Omega \right) \phi(k,\lambda) = 0,
\]

(2)

where, the operators \( L_0, L_1, L_2, \ldots \) may contain \( \partial/\partial k, k, \lambda \) but not \( \partial/\partial \lambda \), and \( \Omega \) is the global eigenvalue associated with the 2-D ballooning equation. The ballooning operator \( L_0 \) embodies the approximate translational symmetry \((x \to x + 1, l \to l + 1)\) of the original 2-D eigenmode equation and is invariant under the combined parity (CP)